

Flatness of Tensor Products and Semi-Rigidity for C_2 -cofinite Vertex Operator Algebras II

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Abstract

Let V be a simple C_2 -cofinite VOA of CFT type and we study the properties of non-semi-simple modules. We assume that there is a V -module Q such that $\text{Hom}_V(Q \boxtimes V', V) \neq 0$. Let us consider a trace function Ψ_V^{tr} on V . As the author has shown in [5], an S -transformation $S(\Psi_V^{\text{tr}})$ of Ψ_V^{tr} corresponding to $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ may contain pseudo-trace functions. In this paper, we assume that $S(\Psi_V^{\text{tr}})$ is a linear combination of trace functions (i.e. no pseudo-trace functions), then we show that all V -modules are semi-rigid and all trace functions Ψ_U^{tr} on simple modules U appear in $S(\Psi_V^{\text{tr}})$. As such an example, we show that a C_2 -cofinite orbifold model V of a rational VOA of CFT type has no pseudo-trace functions in $S(\Psi_V^{\text{tr}})$. As a corollary of our main theorem, such an orbifold model becomes rational.

1 Introduction

Let $V = \bigoplus_{n=0}^{\infty} V_n$ be a simple C_2 -cofinite vertex operator algebra of CFT type (i.e. $\dim V_0 = 1$). Since V is C_2 -cofinite, a fusion product $W \boxtimes U$ of finitely generated V -modules W and U is well-defined as a maximal finitely generated V -module with a surjective intertwining operator of W from U to $W \boxtimes U$. From the maximality, we can induced a canonical homomorphism $\delta \boxtimes \text{id}_U : W \boxtimes U \rightarrow W^1 \boxtimes U$ from a V -homomorphism $\delta : W \rightarrow W^1$, where id_U denotes the identity map on U . We assume:

Hypothesis I: For every irreducible V -module W , there is an irreducible V -module \widetilde{W} such that $\text{Hom}_V(W \boxtimes \widetilde{W}, V) \neq 0$.

Since V is C_2 -cofinite, the associativity of fusion products holds and so Hypothesis I is equivalent to the existence of \widetilde{V}' for a restricted dual V' of V because of

$$(\widetilde{V}' \boxtimes W') \boxtimes W \cong \widetilde{V}' \boxtimes (W \boxtimes W') \xrightarrow{\text{epi}} \widetilde{V}' \boxtimes V' \xrightarrow{\text{epi}} V.$$

In the case $V' \cong V$, then we can take a restricted dual W' of W as \widetilde{W} .

The author introduced a concept of "Semi-Rigidity" in [6]. We call an irreducible V -module W **semi-rigid** if there are epimorphisms $e_W : W \boxtimes \widetilde{W} \rightarrow V$ and $e_{\widetilde{W}} : \widetilde{W} \boxtimes W \rightarrow V$ and a homomorphism $\rho : P \rightarrow \widetilde{W} \boxtimes W$ satisfying $e_{\widetilde{W}}\rho(P) = V$ such that

$$(e_W \boxtimes \text{id}_W)(\mu(\text{id}_W \boxtimes \rho)(W \boxtimes P)) = V \boxtimes W,$$

where $\mu : W \boxtimes (\widetilde{W} \boxtimes W) \rightarrow (W \boxtimes \widetilde{W}) \boxtimes W$ is a canonical isomorphism (see (2.2)), and P is a projective cover of V . Namely, we consider the following diagram

$$\begin{array}{ccc} W \boxtimes \rho(P) & \subseteq & W \boxtimes (\widetilde{W} \boxtimes W) \xrightarrow{\mu} (W \boxtimes \widetilde{W}) \boxtimes W \\ \downarrow \text{id}_W \boxtimes e_{\widetilde{W}} & & \downarrow \\ W \boxtimes V & \cong & W \boxtimes V \qquad \qquad \downarrow e_W \boxtimes \text{id}_W \\ & & V \boxtimes W \end{array}$$

We call V semi-rigid when all simple V -modules are semi-rigid.

As the author showed in [5], if we consider a trace function $\Psi_V^{\text{tr}}(*, \tau)$ on V by

$$\Psi_V^{\text{tr}}(v, \tau) = \text{Tr}_V(o(v)q^{\tau(L(0)-c/24)}) = \sum_{n=0}^{\infty} (\text{Tr}_{V_n} o(v))q^{(n-c/24)\tau} \quad \text{for } v \in V,$$

then its S -transformation $S(\Psi_{V,\text{tr}})$ (corresponding to $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in SL_2(\mathbb{Z})$), which is given by $(\frac{-1}{\tau})^{\text{wt}(v)} \Psi_V^{\text{tr}}(v, -1/\tau)$, equals to a linear combination of trace functions Ψ_U^{tr} on simple modules U and pseudo-trace functions Ψ_T^ϕ , where q^τ denotes $e^{2\pi i\tau}$, c is a central charge of V and $o(v)$ denotes a grade-preserving operator of v , (e.g. $o(v) = v_{m-1}$ for $v \in V_m$). In other words, there are $\lambda_{(U,\text{tr})}, \lambda_{(T,\phi)} \in \mathbb{C}$ such that for $v \in V$ and $0 < |q^\tau| < 1$, we have:

$$S(\Psi_V^{\text{tr}})(v, \tau) = \sum_{U \text{ irr.}} \lambda_{(U,\text{tr})} \Psi_U^{\text{tr}}(v, \tau) + \sum_{(T,\phi), \phi \neq \text{tr}} \lambda_{(T,\phi)} \Psi_T^\phi(v, \tau).$$

Hypothesis II: $S(\Psi_V^{\text{tr}})$ is a linear combination of trace functions on modules.

The aim of this paper is to show the following theorem.

Main Theorem *If V is a simple C_2 -cofinite vertex operator algebra of CFT-type satisfying Hypothesis I and II, then V is semi-rigid and $\lambda_{(U,\text{tr})} \neq 0$ for every simple module U .*

At last, we will show an example satisfying Hypothesis II.

Theorem 4 *Let T be a rational vertex operator algebra of CFT type and τ is a finite automorphism of T . We assume that the fixed point subVOA T^τ is C_2 -cofinite and satisfies Hypothesis I. Then T^τ has no pseudo-trace functions in $S(\Psi_{T^\tau}^{\text{tr}})$.*

Here a VOA T is called rational if all \mathbb{N} -gradable modules are completely reducible. As a corollary, we have:

Corollary 7 *Under the assumption of Theorem 4, T^τ is also rational.*

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2 Preliminary results

2.1 The space of logarithmic intertwining operators

Throughout this paper, we will treat only finitely generated modules and $\text{mod}(V)$ denotes the set of all finitely generated V -modules. We should note that all modules are \mathbb{N} -gradable because V is C_2 -cofinite. If V is also rational, then Proposition 5.1 in [4] implies that V is rigid (which is stronger than semi-rigid). Our aim is to extend his result to non-semi-simple modules. Therefore, our concept of intertwining operators includes logarithmic intertwining operators \mathcal{Y} of type $\binom{A}{U, B}$ which has a form $\mathcal{Y}(u, z) = \sum_{i=0}^K \sum_r u_{r,i} z^{-r-1} \log^i z$ with $u_{r,i} \in \text{Hom}(B, A)$ for $u \in U$. Y^U denotes a vertex operator of V on a module U , and $\mathcal{I}_{A,B}^C$ denotes the space of (logarithmic) intertwining operators of type $\binom{C}{A, B}$ for $A, B, C \in \text{mod}(V)$. We fix one surjective intertwining operator $\mathcal{Y}_{A,B}^\boxtimes \in \mathcal{I}_{A,B}^{A \boxtimes B}$ for each pair A, B . Here "surjective" implies that the images of $\mathcal{Y}_{A,B}^\boxtimes$ are not contained a proper subspace. For $W \in \text{mod}(V)$, W_r denotes a generalized eigenspace $\{w \in W \mid (L(0) - r)^N w = 0 \text{ for some } N \in \mathbb{N}\}$ and W' denotes a restricted dual V -module $\oplus_r \text{Hom}(W_r, \mathbb{C})$ of W .

2.2 Analytic functions

We first recall the analytic part on the composition of intertwining operators (with logarithmic terms) from [3]. From now on, let $A, B, C, D, E, F \in \text{mod}(V)$ and $a \in A, b \in B, c \in C, d' \in D'$. As Huang showed, for intertwining operators $\mathcal{Y}_1 \in \mathcal{I}_{A,E}^D$, $\mathcal{Y}_2 \in \mathcal{I}_{B,C}^E$, $\mathcal{Y}_3 \in \mathcal{I}_{F,C}^D$ and $\mathcal{Y}_4 \in \mathcal{I}_{A,B}^F$, the formal power series (with logarithmic terms)

$$\langle d', \mathcal{Y}_1(a, x) \mathcal{Y}_2(b, y) c \rangle \quad \text{and} \quad \langle d', \mathcal{Y}_3(\mathcal{Y}_4(a, x - y) b, y) c \rangle$$

are absolutely convergent in $\Delta_1 = \{(x, y) \in \mathbb{C}^2 \mid |x| > |y| > 0\}$ and $\Delta_2 = \{(x, y) \in \mathbb{C}^2 \mid |y| > |x - y| > 0\}$, respectively, and can all be analytically extended to multi-valued analytic functions on

$$M^2 = \{(x, y) \in \mathbb{C}^2 \mid xy(x - y) \neq 0\}.$$

As he did, we are able to lift them to single-valued analytic functions

$$E(\langle d, \mathcal{Y}_1(a, x) \mathcal{Y}_2(b, y) c \rangle) \quad \text{and} \quad E(\langle d, \mathcal{Y}_3(\mathcal{Y}_4(a, x - y) b, y) c \rangle)$$

on the universal covering $\widetilde{M^2}$ of M^2 . As he remarked, single-valued liftings are not unique, but the existence of such functions is enough for our arguments. The important fact is that if we fix A, B, C, D , then these functions are given as solutions of the same differential

equations. Therefore, for $\mathcal{Y}_1 \in \mathcal{I}_{A,E}^D$, $\mathcal{Y}_2 \in \mathcal{I}_{B,C}^E$ there are $\mathcal{Y}_5 \in \mathcal{I}_{A \boxtimes B, C}^D$ and $\mathcal{Y}_6 \in \mathcal{I}_{B, A \boxtimes C}^D$ such that

$$\begin{aligned} E(\langle d', \mathcal{Y}_1(a, x) \mathcal{Y}_2(b, y) c \rangle) &= E(\langle d', \mathcal{Y}_5(\mathcal{Y}_{A,B}^{\boxtimes}(a, x-y) b, y) c \rangle) \quad \text{and} \\ E(\langle d', \mathcal{Y}_2(\mathcal{Y}_4(a, x-y) b, y) c \rangle) &= E(\langle d', \mathcal{Y}_6(a, x) \mathcal{Y}_{B,C}^{\boxtimes}(b, y) c \rangle). \end{aligned} \quad (2.1)$$

We note that the right hand sides of (2.1) are usually expressed by linear sums, say,

$$E(\langle d', \mathcal{Y}_1(a, x) \mathcal{Y}_2(b, y) c \rangle) = \sum_i E(\langle d', \mathcal{Y}_{1i}(\mathcal{Y}_{2i}(a, x-y) b, y) c \rangle).$$

For each term, from the maximality of fusion products, there is a homomorphism $\xi_i \in \text{Hom}_V(A \boxtimes B, \text{Im}(\mathcal{Y}_{2i}))$ such that $\mathcal{Y}_{2i} = \xi_i \mathcal{Y}_{A,B}^{\boxtimes}$. Then it is easy to check that $\sum_i \mathcal{Y}_{1i} \xi_i$ is an intertwining operator in $\mathcal{I}_{A \boxtimes B, C}^D$ and so we can get the expressions (2.1). The canonical isomorphism $\mu : (A \boxtimes B) \boxtimes C \rightarrow A \boxtimes (B \boxtimes C)$ is given by

$$E(\langle d', \mu \mathcal{Y}_{A \boxtimes B, C}^{\boxtimes}(\mathcal{Y}_{A,B}^{\boxtimes}(a, x-y) b, y) c \rangle) = E(\langle d', \mathcal{Y}_{A, B \boxtimes C}^{\boxtimes}(a, x) \mathcal{Y}_{B,C}^{\boxtimes}(b, y) c \rangle). \quad (2.2)$$

2.3 Skew symmetric and adjoint intertwining operators

In his paper [4], Huang explicitly defined a skew symmetry intertwining operator $\sigma_{12}(\mathcal{Y}) \in \mathcal{I}_{B,A}^C$ and an adjoint intertwining operator $\sigma_{23}(\mathcal{Y}) \in \mathcal{I}_{A,C'}^{B'}$ for $\mathcal{Y} \in \mathcal{I}_{A,B}^C$ under the assumption that \mathcal{Y} has no logarithmic terms. Even if $\mathcal{Y} \in \mathcal{I}_{B,A}^C$ has logarithmic terms, by considering a path $\{z = \frac{1}{2}e^{\pi i t}x \mid t \in [0, 1]\}$, there is $\tilde{\mathcal{Y}} \in \mathcal{I}_{A,B}^C$ such that

$$E(\langle c', \tilde{\mathcal{Y}}(a, z) \sigma_{12}(Y^B)(b, x) \mathbf{1} \rangle) = E(\langle c', \mathcal{Y}(b, x) \sigma_{12}(Y^A)(a, z) \mathbf{1} \rangle), \quad (2.3)$$

which implies there is an isomorphism $\sigma_{12} : \mathcal{I}_{A,B}^C \cong \mathcal{I}_{B,A}^C$. We rewrite them.

$$\begin{aligned} \text{The left side of (2.3)} &= E(\langle c', \tilde{\mathcal{Y}}(a, z) e^{L(-1)x} b \rangle) = E(\langle c', e^{L(-1)x} \tilde{\mathcal{Y}}(a, z-x) b \rangle) \\ &= E(\langle e^{L(1)x} c', \tilde{\mathcal{Y}}(a, z-x) b \rangle) \\ \text{The right side of (2.3)} &= E(\langle c', \mathcal{Y}(b, x) e^{L(-1)(z)} a \rangle) = E(\langle c', e^{L(-1)z} \mathcal{Y}(b, x-z) a \rangle) \\ &= E(\langle e^{L(1)x} c', e^{L(-1)(z-x)} \mathcal{Y}(b, x-z) a \rangle). \end{aligned}$$

Since $\langle e^{L(1)x} c', \tilde{\mathcal{Y}}(a, z-x) b \rangle$ and $\langle e^{L(1)x} c', e^{L(-1)z} \mathcal{Y}(b, x-z) a \rangle$ are multivalued rational functions on $\{(x, z) \mid x \neq z\}$, we may choose σ_{12} so that

$$\sigma_{12}(\mathcal{Y})(a, z-x) b = e^{L(-1)(z-x)} \mathcal{Y}(b, x-z) a. \quad (2.4)$$

Similarly, for $\mathcal{Y} \in \mathcal{I}_{A,B}^{C'}$ and canonical intertwining operators $\mathcal{Y}_{C,C'}^{V'}$ and $\mathcal{Y}_{B',B}^{V'}$ induced from inner products, there is $\mathcal{Y}^4 \in \mathcal{I}_{A,C}^{B'}$ such that

$$E(\langle \mathbf{1}, \mathcal{Y}_{C,C'}^{V'}(c, x) \mathcal{Y}(a, y) b \rangle) = E(\langle \mathbf{1}, \mathcal{Y}_{B',B}^{V'}(e^{L(-1)(x-y)} \mathcal{Y}^4(a, y-x) c, y) b \rangle).$$

Therefore, we have an isomorphism $\sigma_{23} : \mathcal{I}_{A,B}^C \cong \mathcal{I}_{A,C'}^{B'}$. We need the notation $\sigma_{23}(\mathcal{Y})$, but not an explicit formula in this paper.

In (2.1), we used \mathcal{Y}^{\boxtimes} as the second intertwining operator of products. Not only the second one, we can also use it for the first one at the same time. Actually, for $\mathcal{Y}_5(\mathcal{Y}_{A,B}^{\boxtimes})$

with $\mathcal{Y}_5 \in \mathcal{I}_{A \boxtimes B, C}^D$, we have $\sigma_{123}^{-1}(\mathcal{Y}_5) \in \mathcal{I}_{C, D'}^{(A \boxtimes B)'}$ and so there is $\delta \in \text{Hom}_V(C \boxtimes D', (A \boxtimes B)')$ such that $\sigma_{123}^{-1}(\mathcal{Y}_5) = \delta \mathcal{Y}_{C, D'}^{\boxtimes}$. Therefore we have:

$$\mathcal{Y}_5(\mathcal{Y}_{A, B}^{\boxtimes}) = \sigma_{123}(\delta \mathcal{Y}_{C, D'}^{\boxtimes})(\mathcal{Y}_{A, B}^{\boxtimes}) = \sigma_{123}(\mathcal{Y}_{C, D'}^{\boxtimes})(\delta^* \mathcal{Y}_{A, B}^{\boxtimes}),$$

where $\delta^* \in \text{Hom}_V(A \boxtimes B, (C \boxtimes D')')$ is a dual of δ and σ_{123} denotes $\sigma_{12}\sigma_{23}$.

2.4 Semi-rigidity and intertwining operators

We next describe the semi-rigidity in terms of intertwining operators. For a V -module U , let $\text{rad}^V(U)$ denote the smallest submodule such that $U/\text{rad}^V(U)$ is a direct sum of copies of V . From the definition of semi-rigidity, if W is not semi-rigid, then

$$\mu(W \boxtimes \text{rad}^V(\widetilde{W} \boxtimes W)) + \text{Ker}(e_W \boxtimes \text{id}_W) = (W \boxtimes \widetilde{W}) \boxtimes W, \quad (2.5)$$

for any $e_W : W \boxtimes \widetilde{W} \rightarrow V$, where $\mu : (W \boxtimes \widetilde{W}) \boxtimes W \rightarrow W \boxtimes (\widetilde{W} \boxtimes W)$ is a canonical isomorphism. On the other hand, as we has shown in §2.3, for any $e_W \mathcal{Y}_{W, \widetilde{W}}^{\boxtimes} \in \mathcal{I}_{W, \widetilde{W}}^V$, $w, w^1 \in W$, $\widetilde{w} \in \widetilde{W}$, and $a' \in W'$, there is $\delta \in \text{Hom}_V(W \boxtimes \widetilde{W}, (W \boxtimes W')')$ such that

$$E(\langle a', \sigma_{12}(Y^W)(w, x) e_{\widetilde{W}} \mathcal{Y}_{\widetilde{W}, W}^{\boxtimes}(\widetilde{w}, y) w^1 \rangle) = E(\langle a', \sigma_{123}(\mathcal{Y}_{W, W'}^{\boxtimes})(\delta \mathcal{Y}_{W, \widetilde{W}}^{\boxtimes}(w, x - y) \widetilde{w}, y) w^1 \rangle). \quad (2.6)$$

Therefore, W is not semi-rigid if and only if $\text{Image}(\delta)$ does not have a factor isomorphic to V if and only if $\text{Ker}(\delta) + \text{rad}^V(W \boxtimes \widetilde{W}) = W \boxtimes \widetilde{W}$ for any $e_{\widetilde{W}}$.

2.5 Pseudo-trace

Although we won't treat pseudo-trace functions [5] in this paper, we will explain them a little. It was introduced to explain a symmetric function on n -th Zhu algebra $A_n(V)$ in terms of V -modules. Most all of symmetric functions on $A_n(V)$ are linear combinations of traces of grade-preserving operators $o(v)$ with $v \in V$ on n -th lowest homogeneous weight-space $U(n)$ of simple V -modules U . However, in some VOAs, these functions don't cover all symmetric functions. The remaining symmetric functions are given by the following: For a V -module U with submodules $U \supseteq T \supseteq S$ and a surjective V -homomorphism $\phi : U \rightarrow S$ with $\text{Ker}(\phi) = T$, take a transversal $\epsilon : S \rightarrow U$, that is, $\phi\epsilon = 1_S$ and choose a basis $\{s^i \mid i \in I\}$ of S . We extend $\{s^i, \epsilon(s^i)\}$ to a basis $\{s^i, \epsilon(s^i), \dots \mid i \in I\}$ of U and using this basis, we can express the action of V on U as

$$Y^U(v, z) = \begin{pmatrix} A_{11}(v, z) & A_{12}(v, z) & A_{13}(v, z) \\ O & A_{22}(v, z) & A_{23}(v, z) \\ O & O & A_{11}(v, z) \end{pmatrix}. \quad (2.7)$$

Then a pseudo-trace function on (U, ϕ) is defined by

$$\text{Tr}_U^\phi Y^U(z^{L(0)}v, z) q^{\tau(L(0)-c/24)} := \sum_i \langle (s^i)', Y^U(z^{L(0)}v, z) q^{\tau(L(0)-c/24)} \epsilon(s^i) \rangle, \quad (2.8)$$

where $\{(s^i)' \mid i \in I\}$ is the dual basis of $\{s^i \mid i \in I\}$. In other words, it is a trace function of $A_{13}(z^{L(0)}v, z) q^{\tau(L(0)-c/24)}$. If it is symmetric with respect to V (we call it V -symmetric),

that is, it is symmetric with the grade-preserving actions of V , then we call it pseudo-trace (function). From (2.7), we have that the value of pseudo-trace is zero for an element which acts on U semi-simply. For example, $\text{Tr}_U^\phi Y^U(\mathbf{1}, z) q^{\tau(L(0)-c/24)} = 0$.

3 Geometrically modified module

We quote the theory of composition-invertible power series and their actions on modules for the Virasoro algebra developed in [3]. From now on, q^x denotes $e^{2\pi i x}$ for variables x to simplify the notation. Let A_j ($j = 1, 2, \dots$) be the complex numbers defined by

$$\frac{1}{2\pi i}(q^y - 1) = \left(\exp \left(- \sum_{j=1}^{\infty} A_j y^{j+1} \frac{\partial}{\partial y} \right) \right) y$$

and set

$$\mathcal{U}(q^x) = q^{xL(0)} (2\pi i)^{L(0)} e^{-\sum_{j=1}^{\infty} A_j L(j)}.$$

The important one is $\mathcal{U}(1)$, which satisfies

$$\mathcal{U}(1)\mathcal{Y}(w, x)\mathcal{U}(1)^{-1} = \mathcal{Y}(\mathcal{U}(q^x)w, q^x - 1) = \mathcal{Y}(q^{xL(0)}\mathcal{U}(1)w, q^x - 1) = \mathcal{Y}[\mathcal{U}(1)w, x] \quad (3.1)$$

for an intertwining operator \mathcal{Y} , see [8] for $\mathcal{Y}[\cdot, x]$.

3.1 Trace functions

We first consider q^τ -traces of geometrically-modified module operators with one more variable z :

$$\Psi_U^\phi(v; z, \tau) := \text{Tr}_U^\phi Y(\mathcal{U}(q^z)v, q^z) q^{\tau(L(0)-c/24)} \quad (3.2)$$

for a V -module U and $v \in V$, where Tr_U^ϕ is a pseudo-trace (including an ordinary trace Tr_U) and c is the central charge of V . We note that for an ordinary trace function, we can consider the trace functions for not only V but also a V -module T and $\mathcal{Y} \in \mathcal{I}_{T,U}^U$. Namely, we can define a trace function

$$\Psi_U^{\text{tr}}(\mathcal{Y}; t; z, \tau) := \text{Tr}_U(\mathcal{Y}(\mathcal{U}(q^z)t, q^z) q^{\tau(L(0)-c/24)}) \quad t \in T. \quad (3.3)$$

We have to note that $L(0)$ may not be semisimple on a V -module U . We denote the semisimple part of $L(0)$ by wt and $L(0)^{\text{nil}} = L(0) - \text{wt}$ is a nilpotent part of $L(0)$. Then we will understand $q^{\tau L(0)}$ on U as

$$q^{\tau L(0)} := q^{\tau(\text{wt} + L(0)^{\text{nil}})} = q^{\tau \text{wt}} (e^{2\pi i \tau L(0)^{\text{nil}}}) = q^{\tau \text{wt}} \sum_{j=0}^{\infty} \frac{(2\pi i \tau L(0)^{\text{nil}})^j}{j!}.$$

In particular, trace function may have a term $q^{\tau r} \tau^j$ for $j \in \mathbb{N}$.

We note that for simple modules W and U , $\mathcal{Y}_{W,U}^U \in \mathcal{I}_{W,U}^U$ has no logarithmic terms and the grade-preserving operators $o(w)$ of $w \in W_r$ in $\mathcal{Y}_{W,U}^U(w, z) = \sum w_m z^{-m-1}$ is w_{r-1} . Therefore, by setting $\mathcal{U}(1)w = \sum w^r$ with homogeneous elements $w^r \in W_r$, we have

$$\begin{aligned} \text{Tr}_U^\phi \mathcal{Y}_{W,U}^U(\mathcal{U}(q^z)w, q^z) q^{\tau(L(0)-c/24)} &= \sum_r \text{Tr}_U^\phi q^{z(\text{wt}(w^r))} w_{r-1}^r q^{(-r)} q^{\tau(L(0)-c/24)} \\ &= \sum_r \text{Tr}_U^\phi w_{r-1}^r q^{\tau(L(0)-c/24)}. \end{aligned} \quad (3.4)$$

Thus, (3.4) is independent of z . Moreover, it has shown in [3] that these q^τ -traces are absolutely convergent when $0 < |q^\tau| < 1$ and can be analytically extended to analytic functions of τ in the upper-half plane.

We next consider q^τ -traces of products of two geometrically-modified intertwining operators:

$$\begin{aligned} & \text{Tr}_U^\phi \mathcal{Y}_1(\mathcal{U}(q^y) \mathcal{Y}_{W, \widetilde{W}}^\boxtimes(w, x-y) \widetilde{w}, q^y) q^{\tau(L(0)-c/24)} \\ & \text{Tr}_U^\phi \mathcal{Y}_2(\mathcal{U}(q^x) w, q^x) \mathcal{Y}_{\widetilde{W}, U}^\boxtimes(\mathcal{U}(q^y) \widetilde{w}, q^y) q^{\tau(L(0)-c/24)} \end{aligned} \quad (3.5)$$

for $w \in W, \widetilde{w} \in \widetilde{W}$, $\mathcal{Y}_1 \in \mathcal{I}_{W \boxtimes \widetilde{W}, U}^U$, and $\mathcal{Y}_2 \in \mathcal{I}_{W, \widetilde{W} \boxtimes U}^U$. As we explained, the first function in (3.5) depends on $x-y$, but not on y . These formal power series (with log-terms) are absolutely convergent in $\Omega_1 = \{(x, y, \tau) \in \mathbb{C}^2 \oplus \mathcal{H} \mid 0 < |q^x - q^y| < |q^y|\}$ and $\Omega_2 = \{(x, y, \tau) \in \mathbb{C}^2 \oplus \mathcal{H} \mid 0 < |q^\tau| < |q^y| < |q^x| < 1\}$, respectively, as shown in [3], where $\mathcal{H} = \{\tau \in \mathbb{C} \mid \text{Im}(\tau) > 0\}$ is the upper half plane. We extend these function analytically to multivalued analytic functions on

$$M_1^2 = \{(x, y, \tau) \in \mathbb{C}^2 \times \mathcal{H} \mid x \neq y + p\tau + q \quad \text{for all } p, q \in \mathbb{Z}\}.$$

We can lift them to single valued analytic functions

$$\begin{aligned} \Psi_U^\phi(\mathcal{Y}_1(\mathcal{Y}_{W, \widetilde{W}}^\boxtimes) : w, \widetilde{w}; x, y, \tau) &= E(\text{Tr}_U^\phi \mathcal{Y}_1(\mathcal{U}(q^y) \mathcal{Y}_{W, \widetilde{W}}^\boxtimes(w, x-y) \widetilde{w}, q^y) q^{\tau(L(0)-c/24)}) \\ \Psi_U^\phi(\mathcal{Y}_2 \cdot \mathcal{Y}_{\widetilde{W}, U}^\boxtimes : w, \widetilde{w}; x, y, \tau) &= E(\text{Tr}_U^\phi \mathcal{Y}_2(\mathcal{U}(q^x) w, q^x) \mathcal{Y}_{\widetilde{W}, U}^\boxtimes(\mathcal{U}(q^y) \widetilde{w}, q^y) q^{\tau(L(0)-c/24)}) \end{aligned} \quad (3.6)$$

on the universal covering \widetilde{M}_1^2 . Although Huang has treat only trace functions in [4], but it is still possible for pseudo-trace functions, (see [5]).

We need to extend one statement in [4] to logarithmic intertwining operators.

Lemma 1 *For a (logarithmic) intertwining operator $\mathcal{Y} \in \mathcal{I}_{B, U}^T$, $w \in W$ and $b \in B$, we have*

$$\begin{aligned} e^{\tau L(0)} \mathcal{Y}(b, z) u &= \mathcal{Y}(e^{\tau L(0)} b, e^\tau z) e^{\tau L(0)} u \\ q^{\tau L(0)} \mathcal{Y}(\mathcal{U}(q^y) b, q^y) &= \mathcal{Y}(\mathcal{U}(q^{y+\tau}) b, q^{y+\tau}) q^{\tau L(0)} \quad \text{and} \\ \mathcal{Y}^1(\mathcal{Y}^2(\mathcal{U}(q^y) b, q^y - q^x) \mathcal{U}(q^x) w, q^x) &= \mathcal{Y}^1(\mathcal{U}(q^x) \mathcal{Y}^2(b, y-x) w, q^x) \end{aligned}$$

[Proof] Set $\mathcal{Y}(b, z) = \sum_{h=0}^K \sum_{n \in \mathbb{C}} b_{n,h} z^{-n-1} \log^h z$ and $y = \log z$ to simplify the notation. From $\mathcal{Y}(L(-1)b, z) = \frac{d}{dz} \mathcal{Y}(b, z)$, we have $(L(-1)b)_{n+1,h} = (-n-1)b_{n,h} + (h+1)b_{n-1,h+1}$ and

$$L(0)(b_{n,h} u) - b_{n,h} L(0)u = (L(-1)b)_{n+1,h} + (L(0)b)_{n,h} = (-n-1)b_{n,h} + (h+1)b_{n,h+1} + (L(0)b)_{n,h}$$

for $u \in U$. Therefore, we obtain:

$$\begin{aligned} & e^{\tau L(0)} \left(\sum_{h=0}^K b_{n,h} u y^h \right) e^{(-n-1)y} = \sum_{m=0}^\infty \frac{L(0)\tau}{m!} \left(\sum_h b_{n,h} u y^h e^{(-n-1)y} \right) \\ &= \sum_{m,h,j} \frac{1}{m!} \binom{m}{j} \tau^m (L(0)|_b + L(0)|_u - n - 1)^{m-j} (h+1) \cdots (h+j) b_{n,h+j} u (2\pi i y)^h e^{(-n-1)y} \\ &= \sum_{m,k=0}^\infty \sum_{j=0}^k \frac{1}{(m-j)!} \frac{1}{j!} (\tau(L(0)|_b + L(0)|_u - n - 1))^{m-j} (k-j+1) \cdots (k) \tau^j y^{k-j} b_{n,k} u e^{(-n-1)y} \\ &= \sum_{k=0}^\infty e^{\tau(L(0)|_b + L(0)|_u - (n+1))} b_{n,k} u (y + \tau)^k e^{(-n-1)y} \\ &= \sum_k (e^{\tau L(0)} b)_{n,k} (e^{\tau L(0)} u) e^{(-n-1)(y+\tau)} (y + \tau)^k \\ &= \mathcal{Y}(e^{\tau L(0)} b, e^{\tau+y}) e^{\tau L(0)} u = \mathcal{Y}(e^{\tau L(0)} b, e^\tau z) e^{\tau L(0)} u, \end{aligned}$$

where $L(0)|_b$ and $L(0)|_u$ denote the action of $L(0)$ on b and u , respectively. Replacing τ and y by $2\pi i\tau$ and $2\pi iy$, respectively, we have the second equation. The third comes from $\mathcal{U}(1)\mathcal{Y}(b, x) = \mathcal{Y}(\mathcal{U}(q^x)b, q^x - 1)\mathcal{U}(1)$ and the second equation. ■

4 Transformations

For a C_2 -cofinite VOA V satisfying Hypothesis I and II, $V^{\otimes n}$ is also a C_2 -cofinite VOA satisfying Hypothesis I and II. Moreover, for a V -module W , $W^{\otimes n}$ is a semi-rigid $V^{\otimes n}$ -module if and only if W is semi-rigid. We also have that Ψ_V^{tr} appears in $S(\Psi_V^{\text{tr}})$ if and only if $\Psi_{V^{\otimes n}}^{\text{tr}}$ appears in $S(\Psi_{V^{\otimes n}}^{\text{tr}})$. Therefore, by taking a suitable $V^{\otimes n}$ instead of V , we may assume that W and \widetilde{W} have integer weights to simplify the arguments.

4.1 Three transformations

A (pseudo-)trace function of $\mathcal{Y}^1(\mathcal{Y}^2) \in \mathcal{I}_{E,U}^U(\mathcal{I}_{\widetilde{W},W}^E)$ on U is

$$\Psi_U^\phi(\mathcal{Y}^1(\mathcal{Y}^2) : \widetilde{w}, w; x, y, \tau) = E(\text{Tr}_U^\phi \mathcal{Y}^1(\mathcal{U}(q^y)\mathcal{Y}^2(\widetilde{w}, x - y)w, q^y)q^{\tau(L(0)-c/24)}), \quad (4.1)$$

for $w \in W, \widetilde{w} \in \widetilde{W}$. A modular transformation $S : \tau \rightarrow -1/\tau$ on Ψ_U^ϕ is defined by

$$\begin{aligned} S\left(\Psi_U^\phi\right)(\mathcal{Y}^1(\mathcal{Y}^2) : \widetilde{w}, w; x, y, \tau) \\ = \Psi_U^\phi\left(\mathcal{Y}^1(\mathcal{Y}^2) : \left(\frac{-1}{\tau}\right)^{L(0)}\widetilde{w}, \left(\frac{-1}{\tau}\right)^{L(0)}w; \frac{-1}{\tau}x, \frac{-1}{\tau}y; \frac{-1}{\tau}\right). \end{aligned} \quad (4.2)$$

When $\mathcal{Y}^1(\mathcal{Y}^2) = Y^U(\mathcal{Y})$ for some $\mathcal{Y} \in \mathcal{I}_{\widetilde{W},W}^V$, it has a modular invariance property. In other words, there are $\lambda_{(T,\psi)} \in \mathbb{C}$ such that

$$S\left(\Psi_U^\phi\right)(Y^U(\mathcal{Y}) : \widetilde{w}, w; x, y, \tau) = \sum \lambda_{(T,\psi)} \Psi_T^\psi(Y^T(\mathcal{Y}) : \widetilde{w}, w; x, y, \tau). \quad (4.3)$$

We note that $\lambda_{(T,\psi)}$ does not depend on $\mathcal{Y} \in \mathcal{I}_{\widetilde{W},W}^V$, but on V .

We define actions S, α_t, β_t on R_2^1 by

$$\begin{aligned} (x, y, \tau) &\xrightarrow{S} (-x/\tau, -y/\tau, -1/\tau) \\ \downarrow \beta_t &\qquad\qquad\qquad \downarrow \alpha_t \\ (x, y + t, \tau) &\xrightarrow{S} (-x/\tau, -y/\tau + 1, -1/\tau). \end{aligned} \quad (4.4)$$

Along a line $\mathcal{L} = \{(x, y + t, \tau) \mid t \in [0, 1]\}$ from (x, y, τ) to $(x, y + 1, \tau)$, we define

$$\alpha_t(\Psi_U^\phi)(\mathcal{Y} : \widetilde{w}, w; x, y, \tau) := \Psi_U^\phi(\mathcal{Y} : \widetilde{w}, w; x, y + t, \tau). \quad (4.5)$$

Since $(x, y, \tau) \rightarrow (x, y + t, \tau)$ preserves $\Omega_2 = \{(x, y, \tau) \in \mathbb{C}^2 \oplus H \mid |q^\tau| < |q^y| < |q^x| < 1\}$, we have

$$\begin{aligned} \alpha_t(\Psi_U^\phi)(\mathcal{Y}^1(\mathcal{Y}^2) : \widetilde{w}, w; x, y, \tau) &= \alpha_t(\text{Tr}_U^\phi \mathcal{Y}_3(\mathcal{U}(q^x)\widetilde{w}, q^x)\mathcal{Y}_{\widetilde{W},U}^{\boxtimes}(\mathcal{U}(q^y)w, q^y)q^{\tau(L(0)-c/24)}) \\ &= \text{Tr}_U^\phi \mathcal{Y}^3(\mathcal{U}(q^x)\widetilde{w}, q^x)\mathcal{Y}_{\widetilde{W},U}^{\boxtimes}(\mathcal{U}(q^{y+t})w, q^{y+t})q^{\tau(L(0)-c/24)} \\ &= \text{Tr}_U^\phi \mathcal{Y}^4(\mathcal{U}(q^y)\mathcal{Y}^5(\widetilde{w}, x - y)w, q^y)q^{\tau(L(0)-c/24)} \\ &= \Psi_U^\phi(\mathcal{Y}^4(\mathcal{Y}^5) : \widetilde{w}, w; x, y, \tau) \end{aligned} \quad (4.6)$$

for some \mathcal{Y}^3 and $\mathcal{Y}^4(\mathcal{Y}^5)$, because $\mathcal{Y}_{\widetilde{W},U}^{\boxtimes}(\mathcal{U}(q^{y+t})w, q^{y+t})$ is a linear combination of geometrically modified intertwining operators in $\mathcal{I}_{\widetilde{W},U}^{\boxtimes}$.

An important case is where $U = V$ and $\mathcal{Y}^1(\mathcal{Y}^2) = Y(\mathcal{Y})$ with $\mathcal{Y} \in \mathcal{I}_{\widetilde{W},W}^V$. Then since $W \boxtimes V = W$ is irreducible,

$$\alpha_1(\Psi_V^{\text{tr}})(Y(\mathcal{Y})) = e^{2\pi i \text{wt}(W)} \Psi_V^{\text{tr}}(Y(\mathcal{Y})).$$

We set $\kappa = e^{2\pi i \text{wt}(W)}$. We then define β_t according to a line $S^{-1}(\mathcal{L})$ by

$$\beta_t(\Psi_U^\phi)(\mathcal{Y}^1(\mathcal{Y}^2) : \widetilde{w}, w; x, y, \tau) = \Psi_U^\phi(\mathcal{Y}^1(\mathcal{Y}^2) : \widetilde{w}, w; x, y + t\tau, \tau) \text{ for any } \Psi_U^\phi. \quad (4.7)$$

Since $\alpha S = S\beta$ on R_2^1 and

$$S(\Psi_U^\phi)(\mathcal{Y}^1(\mathcal{Y}^2) : \widetilde{w}, w) = (-1/\tau)^{(\text{wt}(w) + \text{wt}(\widetilde{w}))} \Psi_U^\phi(\mathcal{Y}^1(\mathcal{Y}^2) : \widetilde{w}, w) S,$$

we have the following relation.

Proposition 2

$$\beta_t(S(\Psi_V)) = S(\alpha_t(\Psi_V)). \quad (4.8)$$

For $\mathcal{Y} \in \mathcal{I}_{\widetilde{W},\widetilde{W}}^V$, we have

$$S(\Psi_V^{\text{tr}})(Y(\mathcal{Y})) = \sum_U \lambda_{(U,\text{tr})} \Psi_U^{\text{tr}}(Y(\mathcal{Y}))$$

by Hypothesis II and we will consider the following diagram:

$$\begin{array}{ccc} \Psi_V^{\text{tr}}(Y(\mathcal{Y})) & \xrightarrow{\alpha} & \kappa \Psi_V^{\text{tr}}(Y(\mathcal{Y})) \\ \downarrow S & & \downarrow S \\ \sum \lambda_{(U,\text{tr})} \Psi_U^{\text{tr}}(Y^U(\mathcal{Y})) & \xrightarrow{\beta} & \sum \lambda_{(U,\text{tr})} \beta(\Psi_U^{\text{tr}}(Y^U(\mathcal{Y}))) = \kappa \sum \lambda_{(U,\text{tr})} \Psi_U^{\text{tr}}(Y^U(\mathcal{Y})) \end{array}$$

4.2 The image of β

In this section, we will calculate $\beta_1(\Psi_U^{\text{tr}})(Y^U(\mathcal{Y}_{\widetilde{W},W}^V))$ as a formal power series. In other words, we expand them in the area $0 < |q^y| < |q^x|$ and $0 < |q^\tau| < 1$ as formal (rational) power series of $(x - y)$ and q^τ and τ . We note $|q^{y+t\tau}| \leq |q^y| < |q^x|$.

Set $A = (W \boxtimes U)$ and $\mathcal{Y}_{\widetilde{W},W}^V = e_{\widetilde{W}} \mathcal{Y}_{\widetilde{W},W}^{\boxtimes}$, then we have:

$$\begin{aligned} & \beta_1(\Psi_U^{\text{tr}})(Y^U(\mathcal{Y}_{\widetilde{W},W}^V) : \widetilde{w}, w; x, y, \tau) \\ &= E(\text{Tr}_U Y^U(\mathcal{U}(q^{y+\tau}) \mathcal{Y}_{\widetilde{W},W}^V(\widetilde{w}, x - (y + \tau)) w, q^{y+\tau}) q^{\tau(L(0) - \frac{c}{24})}) \\ &= E(\text{Tr}_U Y^U(\mathcal{Y}_{\widetilde{W},W}^V(\mathcal{U}(q^x) \widetilde{w}, q^x - q^{y+\tau}) \mathcal{U}(q^{y+\tau}) w, q^{y+\tau}) q^{\tau(L(0) - \frac{c}{24})}) \quad \text{by Lemma 1} \\ &= E(\text{Tr}_U \sigma_{23}(\mathcal{Y}_{\widetilde{W},U'}^{\boxtimes})(\mathcal{U}(q^x) \widetilde{w}, q^x) \xi_U \mathcal{Y}_{\widetilde{W},U}^{\boxtimes}(\mathcal{U}(q^{y+\tau}) w, q^{y+\tau}) q^{\tau(L(0) - \frac{c}{24})}) \\ &\quad \text{for some } \xi_U \in \text{Hom}_V(W \boxtimes U, (\widetilde{W} \boxtimes U)') \\ &= E(\text{Tr}_U \sigma_{23}(\mathcal{Y}_{\widetilde{W},U'}^{\boxtimes})(\mathcal{U}(q^x) \widetilde{w}, q^x) q^{\tau(L(0) - \frac{c}{24})} \xi_U \mathcal{Y}_{\widetilde{W},U}^{\boxtimes}(\mathcal{U}(q^y) w, q^y)) \quad \text{by Lemma 1} \\ &= E(\text{Tr}_U \sigma_{23}(\mathcal{Y}_{\widetilde{W},U'}^{\boxtimes})(\mathcal{U}(q^x) \widetilde{w}, q^x) \xi_U q^{\tau(L(0) - \frac{c}{24})} \mathcal{Y}_{\widetilde{W},U}^{\boxtimes}(\mathcal{U}(q^y) w, q^y)) \\ &= E(\text{Tr}_A \mathcal{Y}_{\widetilde{W},U}^{\boxtimes}(\mathcal{U}(q^y) w, q^y) \sigma_{23}(\mathcal{Y}_{\widetilde{W},U'}^{\boxtimes})(\mathcal{U}(q^x) \widetilde{w}, q^x) \xi_U q^{\tau(L(0) - c/24)}) \\ &\quad \text{because the trace is symmetric} \\ &= E(\text{Tr}_A \sigma_{123}(\mathcal{Y}_{A,A'}^{\boxtimes}(\delta_U \mathcal{Y}_{\widetilde{W},\widetilde{W}}^{\boxtimes})(\mathcal{U}(q^y) w, q^y - q^x) \mathcal{U}(q^x) \widetilde{w}, q^x) q^{\tau(L(0) - c/24)}) \\ &\quad \text{for some } \delta_U \in \text{Hom}_V(W \boxtimes \widetilde{W}, (A \boxtimes A')'). \end{aligned}$$

Set $L[-1] = L(-1) + L(0)$ (see [8]). Then we get $U(1)e^{L(-1)z} = e^{(2\pi i)L[-1]z}U(1)$ from (3.1) and so the above equals to the following:

$$\begin{aligned} & E(\text{Tr}_A \sigma_{123}(\mathcal{Y}_{A,A'}^\boxtimes)(\delta_U \mathcal{U}(q^x) \mathcal{Y}_{W,\widetilde{W}}^\boxtimes(w, y-x)\widetilde{w}, q^x) q^{\tau(L(0)-c/24)}) \quad \text{by Lemma 1} \\ &= E(\text{Tr}_A \sigma_{123}(\mathcal{Y}_{A,A'}^\boxtimes)(\delta_U q^{L(0)x} \mathcal{U}(1) e^{L(-1)(y-x)} \sigma_{12}(\mathcal{Y}_{W,\widetilde{W}}^\boxtimes)(\widetilde{w}, x-y)w, q^x) q^{\tau(L(0)-c/24)}) \\ & \quad \text{by skew symmetry intertwining operator, see (2.4)} \\ &= E(\text{Tr}_A \sigma_{123}(\mathcal{Y}_{A,A'}^\boxtimes)(\delta_U q^{L(0)x} e^{(2\pi i)L[-1](y-x)} \mathcal{U}(1) \sigma_{12}(\mathcal{Y}_{W,\widetilde{W}}^\boxtimes)(\widetilde{w}, x-y)w, q^x) q^{\tau(L(0)-c/24)}). \end{aligned}$$

As we explained, the pair of terms $q^{L(0)x}$ and q^x in the above expression is just formal and has no influence. The next term is $e^{(2\pi i)L[-1](y-x)}$. However, since the grade preserving operators of $L[-1]u$ are zero for any $u \in \widetilde{W} \boxtimes W$, we finally have

$$\begin{aligned} & \beta_1(\Psi_U^{\text{tr}})(Y^U(\mathcal{Y}_{\widetilde{W},W}^V) : \widetilde{w}, w; x, y, \tau) \\ &= E(\text{Tr}_A \sigma_{123}(\mathcal{Y}_{A,A'}^\boxtimes) \mathcal{U}(q^x) \delta_U \sigma_{12}(\mathcal{Y}_{W,\widetilde{W}}^\boxtimes))(\widetilde{w}, x-y)w, q^x) q^{\tau(L(0)-c/24)}. \end{aligned} \quad (4.9)$$

In particular, we have the following lemma.

Lemma 3 $\beta_1(\Psi_U^{\text{tr}})(Y^U(\mathcal{Y}_{\widetilde{W},W}^V))$ is again an ordinary trace function.

We express the definitions of ξ_U and δ_U in a short way

$$Y^U(\mathcal{Y}_{\widetilde{W},W}^V) = \sigma_{23}(\mathcal{Y}_{\widetilde{W},U'}^\boxtimes) \xi \mathcal{Y}_{\widetilde{W},U}^\boxtimes \quad \text{and} \quad \mathcal{Y}_{\widetilde{W},U}^\boxtimes \sigma_{23}(\mathcal{Y}_{\widetilde{W},U'}^\boxtimes) \xi_U = \sigma_{123}(\mathcal{Y}_{A,A'}^\boxtimes) (\delta_U \mathcal{Y}_{W,\widetilde{W}}^\boxtimes). \quad (4.10)$$

For $a' \in A'$, $\widetilde{w} \in \widetilde{W}$, $w, w^1 \in W$ and $u \in U$, let us consider

$$\langle a', \mathcal{Y}_{\widetilde{W},U}^\boxtimes(w^1, x) Y^U(e_{\widetilde{W}} \mathcal{Y}_{\widetilde{W},W}^\boxtimes(\widetilde{w}, y-z)w, z)u \rangle \quad (4.11)$$

into two ways. Set $B = \text{Image}(\delta_U)$, then there is $\mathcal{Y}_{B,W}^{(U \boxtimes A')'}$ such that

$$\begin{aligned} (4.11) &= \langle a', \mathcal{Y}_{\widetilde{W},U}^\boxtimes(w^1, x) \sigma_{23}(\mathcal{Y}_{\widetilde{W},U'}^\boxtimes)(\widetilde{w}, y) \xi \mathcal{Y}_{\widetilde{W},U}^\boxtimes(w, z)u \rangle \\ &= \langle a', \sigma_{123}(\mathcal{Y}_{A,A'}^\boxtimes) (\delta_U \mathcal{Y}_{\widetilde{W},W}^\boxtimes(w^1, x-y)\widetilde{w}, y) \mathcal{Y}_{\widetilde{W},U}^\boxtimes(w, z)u \rangle \\ &= \langle a', \sigma_{123}(\mathcal{Y}_{U,A'}^\boxtimes) \mathcal{Y}_{B,W}^{(U \boxtimes A')'} (\delta_U \mathcal{Y}_{\widetilde{W},W}^\boxtimes(w^1, x-y)\widetilde{w}, y-z)w, z)u \rangle. \end{aligned}$$

On the other hand, there is $\mathcal{Y}_{W,V}^W \in \mathcal{I}_{W,V}^W$ and $\epsilon \in \text{Hom}_V(W, (U \boxtimes A')')$ such that

$$\begin{aligned} (4.11) &= \langle a', \mathcal{Y}_{\widetilde{W},U}^\boxtimes(Y_{W,V}^W(w^1, x-z) e_{\widetilde{W}} \mathcal{Y}_{\widetilde{W},W}^\boxtimes(\widetilde{w}, y-z)w, z)u \rangle \\ &= \langle a', \sigma_{123}(\mathcal{Y}_{U,A'}^\boxtimes) (\epsilon Y_{W,V}^W(w^1, x-z) e_{\widetilde{W}} \mathcal{Y}_{\widetilde{W},W}^\boxtimes(\widetilde{w}, y-z)w, z)u \rangle \end{aligned}$$

for any $a' \in A'$ and $u \in U$. We note $\mathcal{Y}_{W,V}^W \in \mathbb{C} \sigma_{12}(Y^W)$. Therefore, we have

$$\epsilon Y_{W,V}^W(w^1, x-z) e_{\widetilde{W}} \mathcal{Y}_{\widetilde{W},W}^\boxtimes(\widetilde{w}, y-z)w = \mathcal{Y}_{B,W}^{(U \boxtimes A')'} (\delta_U \mathcal{Y}_{\widetilde{W},W}^\boxtimes(w^1, x-z)\widetilde{w}, y-z)w.$$

Since the image of ϵ is W , we obtain

$$\epsilon Y_{W,V}^W(w^1, x-z) \mathcal{Y}_{\widetilde{W},W}^\boxtimes(\widetilde{w}, y-z)w = \mathcal{Y}_{B,W}^W (\delta_U \mathcal{Y}_{\widetilde{W},W}^\boxtimes(w^1, x-z)\widetilde{w}, y-z)w$$

for some $\mathcal{Y}_{B,W}^W$. Thus, δ_U in (4.8) essentially coincides with δ in (2.6), which does not depend on the choice of U .

5 Proof of the Main Theorem

We now start the proof of the Main Theorem. Let W be an irreducible module. As we showed in the previous section,

$$\begin{aligned}\beta_1(\sum \lambda_{(U,\text{tr})} \Psi_U^{\text{tr}})(Y(e_{\widetilde{W}} \mathcal{Y}_{\widetilde{W},W}^{\boxtimes})) &= \sum \lambda_{(U,\text{tr})} \beta_1(\Psi_U^{\text{tr}})(Y(e_{\widetilde{W}} \mathcal{Y}_{\widetilde{W},W}^{\boxtimes})) \\ &= \sum \lambda_{(U,\text{tr})} \Psi_{W \boxtimes U}(\mathcal{Y}_{B,U}^U(\delta \mathcal{Y}_{W,\widetilde{W}}^{\boxtimes})).\end{aligned}$$

On the other hand, since $\beta_1(S(\Psi_V)) = S(\alpha_1(\Psi_V))$, we obtain

$$\beta_1(\sum \lambda_{(U,\text{tr})} \Psi_U^{\text{tr}}(Y(e_{\widetilde{W}} \mathcal{Y}_{\widetilde{W},W}^{\boxtimes}))) = \kappa(\sum \lambda_{(U,\text{tr})} \Psi_U^{\text{tr}}(Y(e_{\widetilde{W}} \mathcal{Y}_{\widetilde{W},W}^{\boxtimes}))).$$

Therefore, we have

$$\sum \lambda_{(U,\text{tr})} \Psi_{W \boxtimes U}(\mathcal{Y}_{B,U}^U(\delta \mathcal{Y}_{W,\widetilde{W}}^{\boxtimes})) = \kappa(\sum \lambda_{(U,\text{tr})} \Psi_U^{\text{tr}}(Y(e_{\widetilde{W}} \mathcal{Y}_{\widetilde{W},W}^{\boxtimes}))).$$

Suppose that W is not semi-rigid. As we mentioned, we may assume that a conformal weight $\text{wt}(W)$ of W is an integer. Then $\text{Ker}(\delta) + \text{Ker}(e_{\widetilde{W}}) = \widetilde{W} \boxtimes W$. Set $Q = \text{Ker}(\delta) \cap \text{Ker}(e_{\widetilde{W}})$ and $W \boxtimes \widetilde{W}/Q = Q^1 \oplus Q^2$ with $Q^1 = \text{Ker}(e_{\widetilde{W}})/Q$ and $Q^2 = \text{Ker}(\delta)/Q \cong V$. Then $\Psi_{W \boxtimes U}(\mathcal{Y}_{B,U}^U(\delta \mathcal{Y}_{W,\widetilde{W}}^{\boxtimes}))$ are all given by traces on Q^1 and $\Psi_U^{\text{tr}}(Y(e_{\widetilde{W}} \mathcal{Y}_{\widetilde{W},W}^{\boxtimes}))$ are all given by traces on Q^2 . We hence have

$$\sum \lambda_{(U,\text{tr})} \Psi_{W \boxtimes U}(\mathcal{Y}_{B,U}^U(\delta \mathcal{Y}_{W,\widetilde{W}}^{\boxtimes})) = 0,$$

which contradicts to $\sum \lambda_{(U,\text{tr})} \Psi_U^{\text{tr}}(Y(e_{\widetilde{W}} \mathcal{Y}_{\widetilde{W},W}^{\boxtimes})) \neq 0$. Therefore, W is semi-rigid. Since W is arbitrary, V is semi-rigid.

We next show $\lambda_{(V',\text{tr})} \neq 0$. Choose a simple module U so that $\lambda_{(U,\text{tr})} \neq 0$. Set $W = U'$ and consider the trace function of the $e_{\widetilde{W}} \mathcal{Y}_{\widetilde{W},W}^{\boxtimes}$ in $\beta_1(\Psi_U^{\text{tr}})(e_{\widetilde{W}} \mathcal{Y}_{\widetilde{W},W}^{\boxtimes})$. It has a nonzero scalar multiple of

$$\Psi_{W \boxtimes U}^{\text{tr}}(e_{\widetilde{W}} \mathcal{Y}_{\widetilde{W},W}^{\boxtimes})$$

and so it has a term $\Psi_{V'}^{\text{tr}}(e_{\widetilde{W}} \mathcal{Y}_{\widetilde{W},W}^{\boxtimes})$ with a nonzero coefficient. On the other hand, for any V -modules $T \neq U$, $\beta_1(\Psi_T^{\text{tr}}(e_{\widetilde{W}} \mathcal{Y}_{\widetilde{W},W}^{\boxtimes}))$ has no entries of $\Psi_{V'}^{\text{tr}}(e_{\widetilde{W}} \mathcal{Y}_{\widetilde{W},W}^{\boxtimes})$. Therefore, $\Psi_{V'}^{\text{tr}}(e_{\widetilde{W}} \mathcal{Y}_{\widetilde{W},W}^{\boxtimes})$ has nonzero coefficient in $\beta_1(\sum \lambda_{(U,\text{tr})} \Psi_U^{\text{tr}}(Y(e_{\widetilde{W}} \mathcal{Y}_{\widetilde{W},W}^{\boxtimes})))$.

The remaining thing is to prove $\lambda_{(U,\text{tr})} \neq 0$ for every simple module U . Set $W = U'$. As we showed, $\lambda_{(V',\text{tr})} \neq 0$ and so there is a simple V -module T with $\lambda_{(T,\text{tr})} \neq 0$ such that $\beta_1(\Psi_T^{\text{tr}})(Y^T(\mathcal{Y}_{\widetilde{W},W}^V))$ to have nonzero coefficient at $\Psi_{V'}^{\text{tr}}(Y^U(\mathcal{Y}_{\widetilde{W},W}^V))$. Then since $\text{Hom}_V(T \boxtimes W, V') \neq 0$, $T = (W)' = U$ and so $\lambda_{(U,\text{tr})} \neq 0$ as we desired.

This completes the proof of the Main theorem.

6 Orbifold model

At last, we will show an example satisfying Hypothesis II. Let T be a rational vertex operator algebra of CFT type and $\tau \in \text{Aut}(T)$ of order p . Let $\xi \in \mathbb{C}$ be a primitive p -th

root of unity and decompose T into $T = \bigoplus_{i=0}^{p-1} T^{(i)}$ with $T^{(i)} = \{v \in T \mid \sigma(v) = \xi^i v\}$. We assume that the fixed point subVOA $V := T^\tau$ is C_2 -cofinite and satisfies Hypothesis I.

Theorem 4 *Let T be a rational vertex operator algebra of CFT type and τ is a finite automorphism of T . We assume that the fixed point subVOA T^τ is C_2 -cofinite and satisfies Hypothesis I. Then $S(\Psi_{T^\tau}^{\text{tr}})$ is a linear combination of trace functions.*

Before we start the proof of Theorem 4, we first show the following:

Proposition 5 *Under the assumption in Theorem 4, T^τ is projective as a T^τ -module.*

[Proof] Suppose false and let $0 \rightarrow B \xrightarrow{\epsilon} P \xrightarrow{\phi} T^\tau \rightarrow 0$ be a non-split extension of T^τ . Set $V = T^\tau$. Viewing T as a V -module, we define a fusion product $W = T \boxtimes_V P$ and set $W^{(i)} = T^{(i)} \boxtimes_V P$. We note $W = W^{(0)} \oplus \dots \oplus W^{(n-1)}$ and $W^{(0)} = P$. Similarly, we set $R = (\text{id}_T \boxtimes \epsilon)(T \boxtimes_V B) \subseteq T \boxtimes_V P$ and $R^{(i)} = (\text{id}_{T^{(i)}} \boxtimes \epsilon)(T^{(i)} \boxtimes_V B) \subseteq T^{(i)} \boxtimes_V P$. We note that $(\text{id}_{T^{(i)}} \boxtimes \epsilon)$ may not be injective, but R^i is not zero since there is a canonical epimorphism $\widetilde{T^{(i)} \boxtimes (T^{(i)} \boxtimes P)} \cong (\widetilde{T^{(i)}} \boxtimes T^{(i)}) \boxtimes P \rightarrow V \boxtimes P \cong P$.

As we explained, there is $\mathcal{Y} \in \mathcal{I}_{T,W}^W$ such that

$$E(\langle w', \mathcal{Y}(t, z_1) \mathcal{Y}_{T,P}^\boxtimes(t^1, z_2) p \rangle) = E(\langle w', \mathcal{Y}_{T,P}^\boxtimes(Y(t, z_1 - z_2)t^1, z_2) p \rangle)$$

for $t, t^1 \in T$, $w' \in W'$ and $p \in P$. From the definition of \mathcal{Y} and the Commutativity of vertex operators of T , we have

$$\begin{aligned} E(\langle w', \mathcal{Y}(t^1, z_1) \mathcal{Y}(t^2, z_2) \mathcal{Y}_{T,P}^\boxtimes(t^1, z) p \rangle) &= E(\langle w', \mathcal{Y}(t^1, z_1) \mathcal{Y}_{T,P}^\boxtimes(Y(t^2, z_2 - z)t^3, z) p \rangle) \\ &= E(\langle w', \mathcal{Y}_{T,P}^\boxtimes(Y(t^1, z_1 - z)Y(t^2, z_2 - z)t^3, z) p \rangle) \\ &= E(\langle w', \mathcal{Y}_{T,P}^\boxtimes(Y(t^2, z_2 - z)Y(t^1, z_1 - z)t^3, z) p \rangle) \\ &= E(\langle w', \mathcal{Y}(v^2, z_2) \mathcal{Y}(t^1, z_1) \mathcal{Y}_{T,P}^\boxtimes(t^3, z) p \rangle) \end{aligned}$$

for $t^1, t^2, t^3 \in T$, which implies the Commutativity of $\{\mathcal{Y}(t, z) \mid t \in T\}$. We also have

$$\begin{aligned} E(\langle w', \mathcal{Y}(t^1, z_1) \mathcal{Y}(t^2, z_2) \mathcal{Y}_{T,P}^\boxtimes(t^3, z) p \rangle) &= E(\langle w', \mathcal{Y}_{T,P}^\boxtimes(Y(t^1, z_1 - z)Y(t^2, z_2 - z)t^3, z) p \rangle) \\ &= E(\langle w', \mathcal{Y}_{T,P}^\boxtimes(Y(Y(t^1, z_1 - z_2)t^2, z_2 - z)t^3, z) p \rangle) \\ &= E(\langle w', \mathcal{Y}(Y(t^1, z_1 - z_2)t^2, z_2) \mathcal{Y}_{T,P}^\boxtimes(t^3, z) p \rangle). \end{aligned}$$

Furthermore, taking $t^1 = \mathbf{1}$, we obtain $\mathcal{Y}(t, z)p = \mathcal{Y}_{T,P}^\boxtimes(t, z)p$ for $t \in V, p \in P$ since

$$\begin{aligned} E(\langle w', \mathcal{Y}(t, z_1) p \rangle) &= E(\langle w', \mathcal{Y}(t, z_1) \mathcal{Y}_{T,P}^\boxtimes(\mathbf{1}, z_2) p \rangle) = E(\langle w', \mathcal{Y}_{T,P}^\boxtimes(Y(t, z_1 - z_2)\mathbf{1}, z_2) p \rangle) \\ &= E(\langle w', \mathcal{Y}_{T,P}^\boxtimes(e^{(z_1 - z_2)L(-1)}t, z_2) p \rangle) = E(\langle w', \mathcal{Y}_{T,P}^\boxtimes(t, z_2 + z_1 - z_2) p \rangle) \\ &= E(\langle w', \mathcal{Y}_{T,P}^\boxtimes(t, z_1) p \rangle). \end{aligned}$$

Therefore, $T \boxtimes_V P$ is a T -module and $(\text{id}_V \boxtimes \epsilon)(T \boxtimes_V B)$ is a direct summand of $T \boxtimes_V P$ since T is rational. Then $B = (\text{id}_V \boxtimes \epsilon)(T \boxtimes_V B) \cap T_V^{(0)}P$ is also a direct summand of P as a V -module, which contradicts the choice of P . \blacksquare

We will assert one more general result.

Proposition 6 $T^{(i)}$ is a simple current as a V -module, that is, $T^{(i)} \boxtimes_V D$ is simple for any simple V -module D .

[**Proof**] Set $Q = T \boxtimes_V D$ and $Q^{(i)} = T^{(i)} \boxtimes_V D$. For Q , we will use the same arguments as above. Suppose that $Q^{(i)}$ contains a proper submodule S . Then $S^\perp \cap (Q^{(i)})' \neq 0$ and so we have

$$\begin{aligned} E(\langle d', \mathcal{Y}(t^{(i)}, z_1) \mathcal{Y}(t^{(n-i)}, z) s \rangle) &= E(\langle d', \mathcal{Y}(Y(t^{(i)}, z_1 - z) t^{(n-i)}, z) s \rangle) \\ &= E(\langle d', Y(Y(t^{(i)}, z_1 - z) t^{(n-i)}, z) s \rangle) = 0 \end{aligned}$$

for $t^{(i)} \in T^{(i)}$, $d' \in S^\perp$ and $s \in S$ since $Y(t^{(i)}, z) t^{(n-i)} \in V\{z\}[\log z]$. On the other hand, since $E(\langle q', \mathcal{Y}(t^{(i)}, z_1) \mathcal{Y}(t^{(n-i)}, z) s \rangle) = E(\langle q', \mathcal{Y}(Y(t^{(i)}, z_1 - z) t^{(n-i)}, z) s \rangle) \neq 0$ for some $q' \in (Q^{(i)})'$, $t^{(i)} \in T^{(i)}$ and $t^{(n-i)} \in T^{(n-i)}$, the coefficients in $\{\mathcal{Y}(t^{(n-i)}, z) s \mid s \in S, t^{(n-i)} \in T^{(n-i)}\}$ spans D and so those in $\{\mathcal{Y}(t^{(i)}, z_1) \mathcal{Y}(t^{(n-i)}, z) s \mid t^{(i)} \in T^{(i)}, t^{(n-i)} \in T^{(n-i)}\}$ spans $Q^{(i)}$. Therefore, we have a contradiction. \blacksquare

We now start the proof of Theorem 4. We pick up one twisted simple T -module $M = \bigoplus_{n=0}^\infty M_{\lambda+n/p}$. Then for each i , $W^{(i)} = \bigoplus_{n=0}^\infty M_{\lambda+n+i/p}$ is a simple V -module and we may assume that $T^{(j)} \boxtimes W^{(i)} = W^{(i+j)}$ since $T^{(j)}$ is simple current. Using $W = W^{(0)}$ and \bar{W} , we will consider geometrically modified trace functions. Set $\mathcal{Y} = \mathcal{Y}_{\bar{W}, W}^V$.

Let us consider the images of $\Psi_T(Y(\mathcal{Y}))$ by α_1 and S . Since $W \boxtimes T^{(i)} = W^{(i)}$, $\alpha_1(\Psi_{T^{(i)}}^{\text{tr}})(Y(\mathcal{Y})) = e^{2\pi i \text{wt}(W^{(i)})} \Psi_{T^{(i)}}^{\text{tr}}(Y(\mathcal{Y}))$ by (4.6). Therefore, we have:

$$\alpha_1(\Psi_T^{\text{tr}}(Y(\mathcal{Y}))) = \alpha_1\left(\sum_{i=0}^{p-1} \Psi_{T^{(i)}}^{\text{tr}}(Y(\mathcal{Y}))\right) = e^{2\pi i \text{wt}(W^{(0)})} \left(\sum_{i=0}^{p-1} \xi^i \Psi_{T^{(i)}}^{\text{tr}}(Y(\mathcal{Y}))\right),$$

which coincides with a scalar multiple of a τ -twisted trace function

$$\Psi_T^{\text{tr}}(\tau \cdot Y(\mathcal{Y})) : w, \tilde{w}; x, y, \tau := E(\text{Tr}_T \tau Y^T(\mathcal{U}(q^y) \mathcal{Y}_{\bar{W}, \tilde{W}}^V(w, x - y) \tilde{w}, q^y) q^{\tau(L(0) - c/24)})$$

on T with an action of τ . On the other hand, since T is rational and C_2 -cofinite, $S(\Psi_T^{\text{tr}})$ is a linear combination of trace functions Ψ_U^{tr} on T -modules U which is also a V -module. Therefore, $\beta_1(S(\Psi_T^{\text{tr}}(Y(\mathcal{Y})))) = S(\alpha_1(\Psi_T^{\text{tr}}(Y(\mathcal{Y})))) = e^{2\pi i \text{wt}(W)} S(\Psi_T^{\text{tr}}(\tau \cdot Y(\mathcal{Y})))$ is also a linear combination of trace functions. Since $\Psi_V^{\text{tr}} = \frac{1}{p}(\sum_{i=0}^{p-1} \Psi_T^{\text{tr}}(\tau^i Y(\mathcal{Y})))$, we have the desired conclusion.

This completes the proof of Theorem 4.

Let us go back to the assumptions in Theorem 4. Since $S(\Psi_V^{\text{tr}})$ is a linear combination of trace functions, V satisfies the conditions of the main theorem and so V is semi-rigid. We have also proved that V is projective as a V -module. Therefore, we have the following by Corollary 15 in [5].

Corollary 7 Under the assumptions in Theorem 4, T^τ is rational.

References

- [1] C. Dong, H. Li and G. Mason, *Modular-invariance of trace functions in orbifold theory and generalized Moonshine*, Comm. Math. Phys. 214 (2000), no.1. 1-56.
- [2] I. B. Frenkel, Y.-Z. Huang, J. Lepowsky, *On Axiomatic Approaches to Vertex Operator Algebras and Modules*, Memoirs Amer. Math. Soc. 104, (1993).
- [3] Y.-Z. Huang, *Differential equations, duality and modular invariance*, Commun. Contemp. Math. 7 (2005), no. 5, 649-706.
- [4] Y.-Z. Huang, *Vertex operator algebras and the Verlinde conjecture*, Commun. Contemp. Math. 10 (2008), no. 1, 103-154.
- [5] M. Miyamoto, *Modular invariance of vertex operator algebra satisfying C_2 -cofiniteness*, Duke Math. J. 122 (2004), no. 1, 51-91.
- [6] M. Miyamoto, *Flatness of Tensor Products and Semi-Rigidity for C_2 -cofinite Vertex Operator Algebras I*, arxiv0906.1407, preprint.
- [7] G. Moore and N. Seiberg, *Classical and quantum conformal field theory*, Comm. Math. Phys. 123 (1989), 177-254.
- [8] Y. Zhu, *Modular invariance of characters of vertex operator algebras*, *J. Amer. Math. Soc.* **9** (1996), 237-302.